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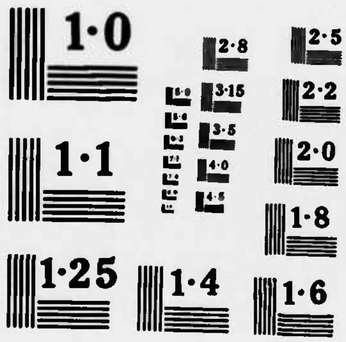

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TWO RESULTS ON DENSE IMBEDDINGS

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TWO RESULTS ON DENSE IMBEDDINGS

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ABSTRACT

In a recent paper [1], J. U. Kim ~~studies~~ studied the Cauchy problem for a Bingham fluid in $\mathbb{R}^2$. R sub 2. He points out that the extension of his results to $\mathbb{R}^3$ sub 3 depends on two lemmas concerning dense imbedding of C_0^∞-functions in certain spaces. In this ~~note~~ paper, these lemmas are proved. C sub 0 infinity

Additional keywords: Sobolev spaces; approximation (mathematics).

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TWO RESULTS ON DENSE IMBEDDINGS

Michael Renardy

Following Kim's notation, we define

$$\tilde{F}_p = \{u \in W^{1,2}(\mathbb{R}^n) \mid \nabla u \in L^p(\mathbb{R}^n)\} \quad .$$

Our first result is the following.

<u>Lemma 1:</u>

$$C_0^\infty(\mathbb{R}^n) \text{ is dense in } \tilde{F}_p \text{ for } 1 \leqq p < \infty \quad .$$

Proof:

Clearly it suffices to show that functions of compact support are dense; C^∞-regularity can easily be achieved by using the Friedrichs mollifier. If we know that $u \in L^p(\mathbb{R}^n)$ or even that $u \in L^{p+\varepsilon}(\mathbb{R}^n)$ for sufficiently small $\varepsilon > 0$, then we can use the standard cut-off procedure to approximate u by functions of compact support. I.e., if we set $u_m(x) = u(x)\psi_m(x)$, where, for example,

$$\psi_m(x) = \begin{cases} 1 & , \ |x| \leqq m \ , \\ 2 - \dfrac{|x|}{m} & , \ m \leqq |x| \leqq 2m \ , \\ 0 & , \ |x| \geqq 2m \ , \end{cases}$$

then it can easily be shown that $u_m \to u$ in $\tilde{F}_p$. Therefore it suffices to show that $\tilde{F}_p \cap L^{p+\varepsilon}$ ($\varepsilon \geqq 0$ small) is dense in $\tilde{F}_p$. If $p \geqq 2$, it follows from the Sobolev imbedding theorem that $\tilde{F}_p \subset L^p$, and there is nothing left to prove.

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In the following, we assume $p < 2$. We set

$$\varphi_N(x) = \begin{cases} \dfrac{1}{N^n \Omega_n} , & |x| \leq N , \\ 0 , & |x| > N . \end{cases}$$

Here Ω_n is the volume of the unit ball in R^n. For $u \in \tilde{F}_p$, let $u_N = u - \varphi_N * u$, where $*$ denotes convolution. Clearly, u_N is in $\tilde{F}_p$, and we want to show $u_N \to u$ in $\tilde{F}_p$. We have $\nabla u_N = \nabla u - \varphi_N * \nabla u$, and, if $p = 1$, then $\int_{R^n} \nabla u = 0$, since $u \in L^2(R^n)$. Therefore, it suffices to show the following: Let $1 \leq r < \infty$, and $v \in L^r$, for $r = 1$, assume in addition that $\int_{R^n} v = 0$. Then $v_N = v - \varphi_N * v \to v$ in L^r.

To see this, note first that $\|\varphi_N\|_{L^1} = 1$, and hence $\|v_N\|_{L^r} \leq 2\|v\|_{L^r}$, hence it suffices to show $v_N \to v$ for v in a dense subset of L^r. If $r > 1$, take $v \in L^1 \cap L^r$. Then $\|\varphi_N * v\|_{L^r} \leq \|\varphi_N\|_{L^r} \|v\|_{L^1}$, which tends to zero. For $r = 1$, let v have compact support, contained in, say $\{|x| \leq R\}$, and assume $\int_{R^n} v = 0$. Then

$$\begin{aligned} \|\varphi_N * v\|_{L^1} &= \int_{R^n} \left| \int_{R^n} \varphi_N(x-y) v(y) dy \right| dx \\ &= \int_{N-R \leq |x| \leq N+R} \left| \int_{|y| \leq R} \varphi_N(x-y) v(y) dy \right| dx \\ &\leq \int_{N-R \leq |x| \leq N+R} \int_{|y| \leq R} |\varphi_N(x-y)| \, |v(y)| dy \, dx \\ &\leq \int_{N-2R \leq |z| \leq N} |\varphi_N(z)| dz \cdot \int_{|y| \leq R} |v(y)| dy . \end{aligned}$$

This tends to zero as $N \to \infty$.

It remains to be shown that u_N lies in $L^{p+\epsilon}(R^n)$ for small $\epsilon > 0$. Let g denote the fundamental solution of the Laplacian. Then an explicit calculation shows that $g - \varphi_N * g$ lies in $L^{1+\delta}(R^n)$ for small positive δ, and so do its derivatives. Hence it follows that $\underline{w}_N := g * \nabla u_N = (g - \varphi_N * g) * \nabla u$ lies in $L^{p+\epsilon}(R^n)$ and so does $u_N = \text{div } \underline{w}_N$. This completes the proof.

Again following Kim's notation, we now define the following spaces of vector-valued functions.

$$G_1(\mathbb{R}^3) = \{\underline{f} \in (W^{1,2}(\mathbb{R}^3))^3 \mid \varepsilon_{ij}(\underline{f}) := \frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i} \in L^1(\mathbb{R}^3)$$

$$\text{for } i,j = 1,2,3, \text{ div } \underline{f} = 0\}$$

$$S(\mathbb{R}^3) = \{\underline{f} \in (C_0^\infty(\mathbb{R}^3))^3 \mid \text{div } \underline{f} = 0\} \quad .$$

The lemma required for Kim's result is the following.

Lemma 2:

$S(\mathbb{R}^3)$ is dense in $G_1(\mathbb{R}^3)$.

Proof:

Obviously, G_1 is contained in

$$G_p = \{\underline{f} \in (W^{1,2}(\mathbb{R}^3)^3 \mid \varepsilon_{ij}(\underline{f}) \in L^p(\mathbb{R}^3), \text{ div } \underline{f} = 0\}$$

for any $p \in [1,2]$. Moreover, lemma 1.9 in [1] says that

$$G_p = F_p = \{\underline{f} \in (W^{1,2}(\mathbb{R}^3))^3 \mid \nabla\underline{f} \in L^p(\mathbb{R}^3), \text{ div } \underline{f} = 0\} \quad .$$

for $p > 1$. Let $\underline{f}_N = \underline{f} - \varphi_N * \underline{f}$ with φ_N as in the proof of lemma 1. Then it follows as before that $\underline{f}_N \to \underline{f}$ in G_1. Let $\underline{a}$ be defined by $\underline{a} = g * \text{curl } \underline{f}$, where g is again the fundamental solution of the Laplacian. The convolution makes sense, since we can write $g = g_1 + g_2$, where $g_1 \in L^1(\mathbb{R}^3)$, $\nabla g_2 \in L^2(\mathbb{R}^3)$, and we define $g_2 * \text{curl } \underline{f}$ by putting the derivative on g_2. Clearly, we have $\text{div } \underline{a} = 0$, $\text{curl } \underline{a} = \underline{f}$. Now let $\underline{a}_N = \underline{a} - \varphi_N * \underline{a}$, so that $\text{curl } \underline{a}_N = \underline{f}_N$, and $\Delta \underline{a}_N = \text{curl } \underline{f}_N$. Since $\underline{a}_N = (g - \varphi_N * g) * \text{curl } \underline{f}$ and $\text{curl } \underline{f} \in L^{1+\varepsilon}(\mathbb{R}^3)$ for ε small, we conclude as in the proof of lemma 1 that $\underline{a}_N \in L^{1+\varepsilon}(\mathbb{R}^3)$. From $\underline{a}_N \in L^{1+\varepsilon}(\mathbb{R}^3)$, $\Delta \underline{a}_N \in L^{1+\varepsilon}(\mathbb{R}^3)$, it follows that $\underline{a}_N \in W^{2,1+\varepsilon}(\mathbb{R}^3)$.

It thus remain to be shown that every $\underline{f} \in G_1$ which has the form $\underline{f} = \operatorname{curl} \underline{a}$ with $\underline{a} \in W^{2,1+\epsilon}(R^3)$ can be approximated by functions of compact support. This is easily achieved by multiplying $\underline{a}$ with an appropriate cut-off function.

REFERENCES

[1] J. U. Kim, On the Cauchy problem associated with the motion of a Bingham fluid in the plane, to appear.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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